

UNIFORM LOCAL ENTROPY FOR ANALYTIC MAPS

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ABSTRACT. Let M be a compact analytic manifold. There is a function $a(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ approaching to 0 as $t \rightarrow 0$ such that, the local entropy $h_{\text{loc}}(f, \varepsilon)$ of any analytic map f on M is upper bounded by the scale $a(\varepsilon)$.

1. INTRODUCTION

In dynamics, a system (f, M) is understood by a continuous map f acting on a compact metric space M . For any compact subset $\Lambda \subset M$, an observable scale $\varepsilon > 0$ and time $n \in \mathbb{N}$, a subset $K \subset \Lambda$ is said (n, ε) -spanning Λ if for any $x \in \Lambda$ there exists $y \in K$ which stays with x in the same scale ε for all time $i \in [0, n)$, i.e., $d(f^i x, f^i y) \leq \varepsilon$, $\forall i \in [0, n)$. Let $r_n(f, \Lambda, \varepsilon)$ denote the smallest cardinality of any (n, ε) -spanning set of Λ . The ε -topological entropy of Λ is defined to be the exponential growth rate of (n, ε) -orbits:

$$h(f, \Lambda, \varepsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n(f, \Lambda, \varepsilon).$$

Let $\varepsilon \rightarrow 0$, we can exhaust all orbits in exponential sense and define the topological entropy of f on Λ by

$$h(f, \Lambda) = \lim_{\varepsilon \rightarrow 0} h(f, \Lambda, \varepsilon).$$

For symbol simplicity, $h(f, \varepsilon) = h(f, M, \varepsilon)$, $h(f) = h(f, M)$.

Given $x \in M$, $n \in \mathbb{N}$, denote the n -step dynamical ball $B_n(f, x, \varepsilon)$ consisting of all such points $y \in M$ that

$$d(f^i y, f^i x) < \varepsilon, \quad i = 0, 1, \dots, n-1.$$

Let $B_\infty(f, x, \varepsilon) = \bigcap_{n \in \mathbb{N}} B_n(f, x, \varepsilon)$. Define the ε -local entropy

$$h_{\text{loc}}(f, \varepsilon) = \sup_{x \in M} h(f, B_\infty(f, x, \varepsilon)).$$

We say f is entropy expansive if there exists $\delta > 0$ such that

$$h_{\text{loc}}(f, \delta) = 0.$$

Comparing with the notion of entropy expansiveness,

(1) as a special case, f is called expansive if there exists $\delta > 0$ such that

$$B_\infty(f, x, \delta) = \{x\}, \quad \forall x \in M;$$

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(2) as a general situation, f is called asymptotically entropy expansive if

$$\lim_{\delta \rightarrow 0} h_{\text{loc}}(f, \delta) = 0.$$

In [1] the entropy expansiveness is proposed by Bowen as a topological condition guaranteeing the upper-semicontinuity of measure theoretic entropy, which together with the variational principle implies the existence of maximal measures. It is asked by Bowen [1] that whether there exist non entropy expansive diffeomorphisms? For any $1 < r < \infty$, Misiurewicz [5] constructed C^r diffeomorphisms without any maximal measure hence not entropy expansive and also not asymptotically entropy expansive. It is revealed from the work of Yomdin [7], Gromov [3], Newhouse [6] that the entropy structure actually depends on the smoothness of dynamical systems. The larger the differential order of dynamics is, the more regular the local entropy is. By contrast with finite differentiability, Buzzi [2] established that all C^∞ maps are asymptotically entropy expansive. With analytic regularity, we can expect more uniform estimation for local entropy and in this content by polynomial approximation and Bernstein inequality, Yomdin [8] has obtained that $h_{\text{loc}}(f, \varepsilon)$ is upper bounded by $\log |\log \varepsilon| / |\log \varepsilon|$ for any surface analytic map. A further question is following

Question 1.1. Is every analytic map entropy expansive?

Remark 1.2. It is known that uniformly hyperbolic systems are expansive¹ (hence entropy expansive). In opposite, there exist analytic and even polynomial maps with non-uniformly hyperbolic behavior which are non-expansive, see Milnor [4].

In present note we plan to study more estimates for the local entropy of analytic maps towards the Question 1.1.

Theorem A. *Let M be a compact analytic manifold. There is a function $a(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ approaching to 0 as $t \rightarrow 0$ such that, for any analytic map f of M there exists $C > 0$ satisfying that for any $\varepsilon > 0$,*

$$h_{\text{loc}}(f, \varepsilon) \leq Ca(\varepsilon).$$

Remark 1.3. Here the function $a(t)$ depends on the dimension of M , independent of the particular f .

Applying Theorem 2.4 of [1], we have the following corollary that gives the approximating rate of ε -entropy $h(f, \varepsilon)$ in the calculation of topological entropy.

Corollary B. *Let M be a compact analytic manifold. There is a function $a(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ approaching to 0 as $t \rightarrow 0$ such that, for any analytic map f of M there exists $C > 0$ satisfying that for any $\varepsilon > 0$,*

$$h(f) - h(f, \varepsilon) \leq Ca(\varepsilon).$$

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¹When f is a homeomorphism, $B_\infty(f, x, \varepsilon) = \{y \in M \mid d(f^i y, f^i x) < \varepsilon, \quad \forall i \in \mathbb{Z}\}$.

2. PROOF OF THEOREM A

Let $m = \dim M$. For any $b > 0$ we denote a standard cube

$$Q_b = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid |x_i| \leq b, 1 \leq i \leq m\}.$$

We say that a map P is C^s ($s \in \mathbb{N}$) on the unit closed cube Q_1 if it is C^s in the interior of Q_1 and all differentials of order up to s can be continuously extended to the boundary of Q_1 . The C^s size of P is

$$\|P\|_s = \sup \{\|d^k P(x)\|, \quad 1 \leq k \leq s, x \in \text{Int } Q_1\}.$$

Fix a positive real number $L_0 > 1$. We first prove Theorem A for analytic map $f : M \rightarrow M$ with $\sup_{x \in M} \|Df_x\| < L_0$.

For f , there is a system of finite local charts $\{(U_1, \gamma_1), \dots, (U_k, \gamma_k)\}$ with $\gamma_i : B(0, 1) \rightarrow U_i$ such that

- if $f(U_i) \cap U_j \neq \emptyset$, then $f_{i,j} := \gamma_j^{-1} \circ f \circ \gamma_i|_{U_i \cap f^{-1}(U_j)}$ can be expressed as a Taylor series.

Such charts are called analytic charts. Noting that $\{U_i \cap f^{-1}(U_j)\}$ constitutes an open cover of M , we denote ρ_0 its Lebesgue number. Thus, by the compactness of M , there is positive number $\rho < \rho_0$ such that the ball $B(x, \rho)$ of any $x \in M$ belongs some $U_i \cap f^{-1}(U_j)$ in which $f_{i,j}$ can be extended to a complex analytic function from $\tilde{B}(\gamma_i^{-1}(x), \rho)$ to \mathbb{C}^m , where $\tilde{B}(\gamma_i^{-1}(x), \rho) = \{y \in \mathbb{C}^m \mid |y - \gamma_i^{-1}(x)| < \rho\}$. We say $\rho = \rho(f)$ is the analytic radius of f . Without loss of generality, suppose $\rho < 1$. Define

$$M_0 = \sup_{i,j,x} |f_{i,j}|_{\tilde{B}(\gamma_i^{-1}(x), \rho)} < \infty.$$

For the sake of statements, we consider $f_{i,j}$ as f in the scale of local analytic charts. Fix an integer n . Since $\sup_{x \in M} \|Df_x\| \leq L_0$, then

$$f^i(B(x, L_0^{-n} \rho)) \subset B(f^i(x), \rho), \quad 0 \leq i \leq n.$$

By Cauchy's formula, for $y \in \text{Int}(D(x, \frac{1}{\sqrt{m}} \rho L_0^{-n})) \subset B(x, \rho L_0^{-n})$, adopting multi-index, for $\alpha = (\alpha_1, \dots, \alpha_m)$ ($\alpha_i \in \mathbb{Z}$ and $\alpha_i \geq 0$),

$$\partial^\alpha f^n(y) = \frac{\alpha!}{(2\pi i)^m} \int_{D(x, \frac{1}{\sqrt{m}} \rho L_0^{-n})} \frac{f^n(z)}{(z_1 - y_1)^{\alpha_1+1} \dots (z_m - y_m)^{\alpha_m+1}} dz,$$

where $D(z, b) = D_1(z_1, b) \times \dots \times D_m(z_m, b)$, $D_i(z_i, b)$ is the circle centered at z_i with radius b . In particular, when $y \in B(x, \frac{1}{2\sqrt{m}} \rho L_0^{-n})$,

$$d(y, D_i(x, \frac{1}{\sqrt{m}} \rho L_0^{-n})) \geq \frac{1}{2\sqrt{m}} \rho L_0^{-n}, \quad 1 \leq i \leq m.$$

Therefore,

$$\begin{aligned} |\partial^\alpha f^n(y)| &\leq \frac{\alpha!}{(2\pi)^m} \left(\frac{1}{2\sqrt{m}} \rho L_0^{-n}\right)^{-|\alpha|-m} \left(2\pi \frac{1}{\sqrt{m}} \rho L_0^{-n}\right)^m M_0 \\ &= \alpha! \cdot 2^m \cdot M_0 \cdot \left(\frac{2\sqrt{m} L_0^n}{\rho}\right)^{|\alpha|}. \end{aligned}$$

Applying Stirling's approximation, for $|\alpha| \geq 1$ we have

$$\begin{aligned}
 (1) \quad |\partial^\alpha f^n(y)| &\leq |\alpha|^{|\alpha|+\frac{1}{2}} e^{1-|\alpha|} \cdot 2^m \cdot M_0 \cdot \left(\frac{2\sqrt{m}L_0^n}{\rho}\right)^{|\alpha|} \\
 &\leq |\alpha|^{|\alpha|+\frac{1}{2}} \cdot 2^m \cdot M_0 \cdot \left(\frac{2\sqrt{m}L_0^n}{\rho}\right)^{|\alpha|}.
 \end{aligned}$$

Take $s_1 = L_0^n n^2$. Further denote $g = f^n$, $\delta = \delta(n) = s_1^{-1}(n)$. There exists $N \geq 3$ such that for any $n \geq N$,

$$(2) \quad \delta(n) \leq \min\{(4\sqrt{m}L_0^n \rho^{-1}n)^{-1}, \rho L_0^{-n}\} \quad \text{and} \quad \frac{\log(4\sqrt{m}\rho^{-1}n^2)}{n} < 1.$$

Along the orbit of x , for $i \in \mathbb{N}$ we define $g_i : B(0, 2) \rightarrow \mathbb{R}^m$ by

$$g_i(t) = s_1(g(s_1^{-1}t + g^{(i-1)}(x)) - g^i(x)).$$

Lemma 2.1. *There is a constant C_0 depends on f , independent of n , such that*

$$\max_{t \in B(0,2)} \|d^k g_i(t)\| \leq C_0 L_0^n, \quad 1 \leq k \leq n, \quad \forall i \in \mathbb{N}, \quad \forall n \geq 1.$$

Proof. Note that

$$\max_{t \in B(0,2)} \|d^k g_i(t)\| \leq s_1^{-k+1} \max_{y \in B(f^{n(i-1)}(x), 2s_1^{-1})} \|d^k f^n(y)\|.$$

We only need to calculate the bound for $n \geq N$. By (1)(2), for $1 \leq k \leq n$ and $n \geq N$, we have

$$\begin{aligned}
 \max_{t \in B(0,2)} \|d^k g_i(t)\| &\leq \left[\left(\frac{2\sqrt{m}}{\rho} L_0^n\right)^{-k+1} (2n)^{-k+1}\right] \cdot [2^m \cdot M_0 \cdot k^{k+\frac{1}{2}} \left(\frac{2\sqrt{m}}{\rho} L_0^n\right)^k] \\
 &\leq \frac{2^{m+1} \sqrt{m} M_0}{\rho} L_0^n \cdot (2n)^{-k+1} k^{k+\frac{1}{2}}.
 \end{aligned}$$

Considering the function $q(k) = (2n)^{-k+1} k^{k+\frac{1}{2}}$, by computation, when $n \geq N \geq 3$,

$$\frac{d}{dk} \big|_{k=1} \log q(k) < 0, \quad \frac{d}{dk} \big|_{k=n} \log q(k) > 0, \quad \frac{d^2}{dk^2} \log q(k) \geq 0, \quad 1 \leq k \leq n.$$

Hence,

$$\max_{1 \leq k \leq n} q(k) = \max\{q(1), q(n)\} \leq \max\left\{1, \frac{n^{\frac{3}{2}}}{2^{n-1}}\right\},$$

which implies

$$\sup_{n \geq N} \max_{1 \leq k \leq n} q(k) < \infty.$$

So, there is a constant $C_0 = C_0(m, \rho, M_0)$ depending on f , independent of n , such that

$$\max_{t \in B(0,2)} \|d^k g_i(t)\| \leq C_0 L_0^n, \quad 1 \leq k \leq n, \quad i \in \mathbb{N}, \quad \forall n \geq 1.$$

□

Given $p \in \mathbb{N}$, denote $F_i = g_i \circ \cdots \circ g_1$, $i = 1, \dots, p$, and let

$$L_i = \{t \in \mathbb{R}^m \mid F_j(t) \in B(0, 1), 0 \leq j \leq i\}.$$

It follows that

$$g^i(B_p(g, x, \delta)) \subset \delta F_i(L_i) + g^i(x), \quad \forall 1 \leq i \leq p.$$

For any $\tau \in \mathbb{N}$, $v = (i_1, \dots, i_m) \in \mathbb{Z}^m$, we define an affine transformation

$$w_{\tau, v} : Q_1 \rightarrow \mathbb{R}^m, \quad z \rightarrow (z + v)/\tau.$$

Then the ball $B(x, \delta)$ is covered by at most $\xi_\delta := ([2\delta/\tau] + 2)^m$ subcubes $w_{\tau, v}(Q_1)$. For each that subcube, we denote $\sigma_v(z) = \delta^{-1}(w_{\tau, v}(z) - x)$. Choose $\tau > 1$ large, we can suppose $\sigma_v(Q_1) \subset B(0, 2)$ and then

$$d^k(\sigma_v) \leq \delta^{-1}\tau^{-1} < 1, \quad 1 \leq k < \infty.$$

Denote $C_n = C_0 L_0^n$.

Proposition 2.2. *For each σ_v and $s \in [1, n]$, there exists a family of C^s maps $\{\psi_{p, j} : Q_1 \rightarrow Q_1, \quad 1 \leq j \leq \kappa^p, p \in \mathbb{N}\}$ with $\kappa = \kappa(s, m, C_n) = \mu(s, m)(\log C_n)^{\nu(s, m)} C_n^{\frac{2m}{s}}$ satisfying the following properties:*

- $F_p(L_p) \subset \cup_{1 \leq j \leq \kappa^p} F_p \circ \sigma_v \circ \psi_{p, j}(Q_1) \subset B(0, 2)$;
- $\|\psi_{p, j}\|_s \leq 1$;
- $\max_{t \in Q_1} \|F_p \circ \sigma_v \circ \psi_{p, j}(t)\|_s \leq 1, j = 1, \dots, \kappa^p$;
- For any $p \in \mathbb{N}$, $i \in \{1, \dots, \kappa^p\}$ there exists $j \in \{1, \dots, \kappa^{p-1}\}$ and a map $\phi_{p, i}^{p-1, j}$ with $\|\phi_{p, i}^{p-1, j}\|_s \leq 1$ such that

$$\psi_{p, i} = \psi_{p-1, j} \circ \phi_{p, i}^{p-1, j}.$$

The constants $\mu = \mu(s, m)$ and $\nu = \nu(s, m)$ depend only on s and m , independent of n .

The above proposition is Lemma 2.3 of Yomdin [7] with the additional contracting properties in the norm $\|d^k \cdot\|$ as observed by Buzzi [2].

For $n \in \mathbb{N}$, we can choose $s(n) \in [1, n]$ such that

$$(3) \quad s(1) \leq s(2) \leq \dots \leq s(n) \leq \dots, \quad s(n+1) - s(n) \leq 1, \quad \lim_{n \rightarrow \infty} s(n) = +\infty,$$

$$(4) \quad \lim_{n \rightarrow \infty} \frac{\mu(s(n), m+1)}{n^{1/4}} = \lim_{n \rightarrow \infty} \frac{\nu(s(n), m+1)}{n^{1/4}} = 0.$$

Noting that $\mu(s(n), m) \leq \mu(s(n), m+1)$, $\nu(s(n), m) \leq \nu(s(n), m+1)$, it also holds that

$$(5) \quad \lim_{n \rightarrow \infty} \frac{\mu(s(n), m)}{n^{1/4}} = \lim_{n \rightarrow \infty} \frac{\nu(s(n), m)}{n^{1/4}} = 0.$$

Given $\varepsilon_1 > 0$, take ε small so that

$$(6) \quad d(f^i x_1, f^i x_2) < \varepsilon_1, \quad \forall i = 0, \dots, n-1, \quad \forall d(x_1, x_2) < \varepsilon.$$

Now fix L to be one ε -dense subset of Q_1 . Let

$$R = x + \delta(\cup_{1 \leq i \leq \kappa^{p-1}} \sigma_v \circ \psi_{p-1, i}(L)).$$

We claim that R is a (p, ε) -spanning set of g restricted to $B_p(g, x, \delta)$. Indeed, for any $y \in B_p(g, x, \delta)$ there exists $t \in Q_1$ and $i \in \{1, \dots, \kappa^{p-1}\}$ such that

$$y = \delta\sigma_v \circ \psi_{p-1,i}(t) + x.$$

We can choose $b \in L$ satisfying $|t - b| \leq \varepsilon$. Denote $b' = \delta\sigma_v \circ \psi_{p-1,i}(b) + x$. Then for every $q = 0, \dots, p-1$, we have

$$g^q(y) = g^q(\delta\sigma_v \circ \psi_{p-1,i}(b) + x) = \delta F_q \circ \sigma_v \circ \psi_{p-1,i}(b) + g^q(x).$$

Notice that $\psi_{p-1,i} = \psi_{q,j} \circ \phi_{p-1,i}^{q,j}$ for some j . Moreover, the maps $\phi_{p-1,i}^{q,j}$ and $\delta F_q \circ \sigma_v \circ \psi_{q,j}$ are contracting in the norm $\|\cdot\|_s$. Thus,

$$|g^q(y) - g^q(b')| \leq |t - b| \leq \varepsilon.$$

Thus R is a (p, ε) -spanning set of g restricted to $B_p(g, x, \delta)$. Observing that the choice of σ_v has ξ_δ , we can deduce that

$$r_p(g, B_p(g, x, \delta), \varepsilon) \leq \xi_\delta \# R \leq \xi_\delta \kappa^{p-1} \# L = \xi_\delta (\mu(\log C_n)^\nu (C_n)^{\frac{2m}{s}})^{p-1} \# L.$$

Since $B_{pn}(f, x, \delta) \subset B_p(f^n, x, \delta) = B_p(g, x, \delta)$, so

$$r_p(f^n, B_{pn}(f, x, \delta), \varepsilon) \leq r_p(f^n, B_p(f^n, x, \delta), \varepsilon/2).$$

Moreover, (6) yields that $r_{pn}(f, B_{pn}(f, x, \delta), \varepsilon_1) \leq r_p(f^n, B_{pn}(f, x, \delta), \varepsilon)$. Thus,

$$\begin{aligned} h(f, B_\infty(f, x, \delta), \varepsilon_1) &= \limsup_{p \rightarrow \infty} \frac{1}{pn} \log r_{pn}(f, B_\infty(f, x, \delta), \varepsilon_1) \\ &\leq \limsup_{p \rightarrow +\infty} \frac{1}{pn} \log r_p(f^n, B_p(f^n, x, \delta), \varepsilon/2) \\ &\leq \frac{1}{n} \left(\frac{2m}{s} \log C_n + \nu(s, m) \log \log C_n + \log \mu(s, m) \right), \end{aligned}$$

which together with the arbitrariness of ε_1 and x gives rise to that

$$(7) \quad h_{\text{loc}}(f, \delta) \leq \frac{1}{n} \left(\frac{2m}{s} \log C_n + \nu(s, m) \log \log C_n + \log \mu(s, m) \right).$$

In addition, $C_n = C_0 L_0^n$. So,

$$\begin{aligned} h_{\text{loc}}(f, \delta) &\leq \frac{2m}{s} \left(\frac{1}{n} \log C_0 + \log L_0 \right) \\ &\quad + \frac{1}{n} (\nu(s, m) \log(n \log L_0 + \log C_0) + \log \mu(s, m)) \\ (8) \quad &\leq \frac{2m}{s} (\log C_0 + \log L_0) \\ &\quad + \frac{1}{n^{1/2}} \left(\frac{1}{n^{1/4}} \nu(s, m) \cdot \frac{1}{n^{1/4}} \log(n \log L_0 + \log C_0) + \frac{1}{n^{1/2}} \log \mu(s, m) \right). \end{aligned}$$

Noting that s is a function relying on n and n can be considered as a function of δ , thus $s = \bar{s}(\delta)$ is a function with variable δ . Since $\delta(n) = (L_0^n n^2)^{-1}$, by (2) for sufficiently large n ,

$$\frac{1}{n} \leq \frac{\log(eL_0)}{-\log \delta(n)}.$$

Define $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as follows

$$a(t) = \begin{cases} 1, & t > \delta(1); \\ \frac{1}{s(\delta(n))} + \frac{1}{(-\log \delta(n))^{1/2}}, & \delta(n+1) < t \leq \delta(n). \end{cases}$$

Lemma 2.3. *For any analytic map \tilde{f} of \tilde{M} with $\dim \tilde{M} = m$ or $m+1$, satisfying*

$$\tilde{L} = \sup_{x \in \tilde{M}} \|D\tilde{f}_x\| < L_0,$$

we have

$$h_{\text{loc}}(\tilde{f}, \delta) \leq C_1 a(\delta),$$

where C_1 is a constant depending on \tilde{f} .

Proof. In this case, there is $N_1 \in \mathbb{N}$ such that for $n \geq N_1$,

$$\delta(n) = n^{-2} L_0^{-n} = n^{-2} L_1^{-n \frac{\log L_0}{\log L}} \leq \rho(\tilde{f}) \tilde{L}^{-n}.$$

Then the estimate (8) applies for sufficiently large n ,

$$\begin{aligned} h_{\text{loc}}(\tilde{f}, \delta(n)) &\leq h_{\text{loc}}(\tilde{f}, \rho(\tilde{f}) \tilde{L}^{-n}) \\ &\leq \frac{2(m+1)}{s(n)} (\log C_0(\tilde{f}) + \log \tilde{L}) \\ &\quad + \frac{1}{n^{1/2}} \left(\frac{1}{n^{1/4}} \nu(s, m+1) \cdot \frac{1}{n^{1/4}} \log(n \log \tilde{L} + \log C_0(\tilde{f})) + \frac{1}{n^{1/2}} \log \mu(s, m+1) \right) \\ &\leq C_1 a(\delta(n)) \quad (\text{by (4)}) \end{aligned}$$

where C_1 is a constant depending on \tilde{f} . □

Next we prove Theorem A for any analytic map \tilde{f} of M . We consider the product $M_1 = \mathbb{R} \times M$. There is a simple flow $\phi(t, (s, x)) = (t+s, x)$ on M_1 called the horizontal flow. We can obtain a suspending manifold \tilde{M} by identifying the points $(t+1, x)$ with $(t, \tilde{f}(x))$. That is, we define an equivalence relation \sim in M_1 by $(t, x) \sim (t_0, x_0)$ iff $t_0 = t+n$ and $x_0 = \tilde{f}^n(x)$. The quotient space $\tilde{M} = M_1 / \sim$ is a smooth manifold and $\dim \tilde{M} = \dim M + 1$. The horizontal flow ϕ pushes down to an analytic flow ψ on \tilde{M} . We can choose $i \in \mathbb{N}$ such that $\psi^{1/i}$ satisfies

$$\sup_{z \in \tilde{M}} \|D\psi_z^{1/i}\| < L_0.$$

Then by Lemma 2.3 we have

$$h_{\text{loc}}(\psi^{1/i}, \delta) \leq C_2 a(\delta),$$

where C_2 is a constant depending on $\psi^{1/i}$. The time one map ψ^1 of ψ on the section $\{0\} \times M$ is smooth conjugate to \tilde{f} . Therefore,

$$(9) \quad h_{\text{loc}}(\tilde{f}, \delta) \leq h_{\text{loc}}(\psi^1, \delta) \leq i h_{\text{loc}}(\psi^{1/i}, \delta L_0^i) \leq i C_2 a(\delta L_0^i).$$

Noticing that $\delta(n) = n^{-2} L_0^{-n}$, so

$$\delta(n) L_0^i = n^{-2} L_0^{-n+i} \leq \delta(n-i).$$

Moreover, $0 \leq s(n) - s(n-i) \leq i$ and $\lim_{n \rightarrow \infty} s(n) = \infty$. So, there exists C_3 such that

$$\begin{aligned} \frac{a(\delta(n)L_0^i)}{a(\delta(n))} &\leq \frac{a(\delta(n-i))}{a(\delta(n))} = \frac{\frac{1}{s(n-i)} + \frac{1}{(-\log \delta(n-i))^{1/2}}}{\frac{1}{s(n)} + \frac{1}{(-\log \delta(n))^{1/2}}} \\ &\leq \frac{\frac{1}{s(n)-i} + \frac{1}{(\log((n-i)^{-2}L_0^{n-i}))^{1/2}}}{\frac{1}{s(n)} + \frac{1}{(\log(n^{-2}L_0^n))^{1/2}}} \\ &\leq C_3, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Therefore, from (9) we deduce that

$$h_{\text{loc}}(\tilde{f}, \delta(n)) \leq iC_2C_3a(\delta(n)).$$

We complete the proof of Theorem A. □

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